# Chapter Two: Understanding

***Understanding*** *tools facilitate the development of students’ mathematical reasoning, analysis, and justification skills, which are needed to communicate understanding in a variety of ways.*

**Understanding Math Students…**

* ***Want to*** understand why the math they learn works.
* ***Like math problems that*** ask them to explain, prove, or take a position.
* ***Approach problem solving*** by looking for patterns and identifying hidden questions.
* ***May experience difficulty when*** there is a focus on the social environment of the classroom (e.g., on collaboration and cooperative problem solving).
* ***Learn best when*** they are challenged to think and explain their thinking.

 (Silver, Thomas, & Perini, 2008)

# [INSERT FIGURE 2.0: UNDERSTANDING MATRIX]

# Always-Sometimes-Never (ASN)

**[NOTE: Set FAC as Sidebar]**

**FACs:** Asking students to consider whether a statement is more than merely “true” or “false” reveals the depth and nuance of students’ understanding.

## [NOTE: Set TechConnect as Sidebar]

## TechConnect: Consider having students use dynamic geometry software to examine changing conditions and test special cases.

## Purpose

Always-Sometimes-Never (ASN) is a reasoning activity that focuses students’ thinking around the important, and often subtle, facts and details associated with mathematical concepts. Students are asked to consider statements containing mathematical information and determine if what is stated is always, sometimes, or never true.

## Overview

The ASN technique has students justify a statement as being *always true,* *sometimes true,* or *never true.* The teacher generates a series of statements for a particular mathematical concept, relates relevant background information or identifies parameters, and asks students to decide if each statement is always, sometimes, or never true. To ensure that students are thinking critically and not just making arbitrary choices, the teacher asks students to explain the reasoning behind their choices. Disagreements and discussions among students are encouraged. ASN can be used to help students:

* Tease out the critical attributes of mathematical concepts. For example, these two statements help students identify the critical similarities and differences between rectangles and squares:
* A rectangle is a square. (Sometimes)
* A square is a rectangle. (Always)
* Recognize the restrictions on a function’s domain or range:
* The equation of a vertical line can be expressed in slope-intercept form. (Never; vertical lines have undefined slope.)
* The domain of a square root function is the set of all non-negative real numbers. (Sometimes; true for *f(x) =* √*x* when *x* ³ *0,* false for *g(x) =* √5–*x* when *x* < *5.*)
* Identify exceptions to mathematical rules. For example:
* *a/a= 1* (Sometimes; the statement is not true for a = 0.)
* Determine the applicability of a theorem or procedure to a given situation. For example:
* If the sides of a triangle are *a*, *b*, and *c* then *a*2 + *b*2 = *c*2*.* (Sometimes; true only when the triangle is a right triangle and *c* is the length of its hypotenuse.)

**Building Common Core Thinking**

Always-Sometimes-Never develops students’ abilities to analyze and explain subtleties, constraints, or exceptions residing in mathematical ideas. Always-Sometimes-Never supports the following Standards for Mathematical Practice (MP):

* ***(MP 1) Sense*** – analyzing givens and constraints and trying special cases.
* ***(MP 2) Reason*** – asking subtle questions to test the reasoning of competing statements
* ***(MP 3) Argument*** – exploring the truth of conjectures, recognizing and using counterexamples, and supporting and justifying conclusions,
* ***(MP 6) Precision*** – formulating careful explanations

## Steps

1. Provide students with a list of statements about a recently discussed or familiar mathematical concept or topic.

2. Allow students enough time to read and consider all of the statements carefully.

3. Have students think about each of the statements and decide whether each is *always true,* *sometimes true,* or *never true.*

4. Make sure that students explain the reasoning behind their choices.

## Examples

### Arithmetic: Addition and Subtraction

1. The sum of two 3-digit numbers is a 3-digit number. (Sometimes)

2. The sum of two even numbers is an odd number. (Never)

3. The difference of two odd numbers is an even number. (Always)

4. The sum of additive inverses is zero. (Always)

5. The difference of three odd numbers is an odd number. (Always)

6. The sum of three even numbers is zero. (Sometimes)

7. The sum of three odd numbers is zero. (Never)

8. The sum of two counting numbers is greater than the difference of the same numbers. (Always)

### Statistics: Mean, Median, Mode

1. A list of numbers has a mean. (Always)

2. A list of numbers has a median. (Always)

3. A list of numbers has a mode. (Sometimes)

4. The mean of a set of numbers is one of the numbers of that set. (Sometimes)

5. The median of ten consecutive integers is one of those integers. (Never)

6. If the mode of a set of numbers is 14, then 14 is one of the numbers of that set. (Always)

7. The mean of a set of numbers is greater than the median of that set of numbers. (Sometimes)

8. The mode of a set of numbers, without repeated values, can be found by arranging the numbers in increasing order and then calculating the mean of the middle two numbers. (Never)

### Trigonometry: Graph Analysis

1. The graph of a trigonometric function is periodic. (Always)

2. Doubling the amplitude of a trigonometric function doubles the period of the function. (Never)

3. The graph of a cosecant function has an infinite number of asymptotes. (Always)

4. The period of *y =* sin *b x + h* is equivalent to the period of *y =* sec *b x + k.* (Always; both )

5. A cosine function has both a maximum and a minimum value. (Always)

6. For numbers *a* and *b,* the graph of *y* = cos *x* on the interval *a* < 0 < *b* is an increasing function. (Never)

7. Stretching the graph of a trigonometric function changes the period of the function. (Sometimes; true for horizontal stretches, false for vertical stretches.)

8. Applying a phase shift on a secant graph changes the location of vertical asymptotes. (Sometimes)

9. For any value of *a,* the graph of *y* = tan *ax* will have the *y*-axis as an asymptote. (Never)

10. Secant graphs have horizontal asymptotes. (Never)

# Compare & Contrast

 **[NOTE: Set note/sentence in a smaller font beneath tool title]**

***Strategy Note:*** For a discussion of the Compare & Contrast strategy, please reference *Styles and Strategies for Teaching Middle School Mathematics* (Thomas & Brunsting, 2010, pp. 60-69) or *Styles and Strategies for Teaching High School Mathematics* (Thomas, Brunsting, & Warrick, 2010, pp. 66-76).

## Purpose

Research shows that teaching students how to identify similarities and differences is one of the most effective ways to increase understanding and raise achievement levels (Marzano, Pickering, & Pollock, 2001). Setting two concepts against one another and using each as a frame of reference for examining the other allows students to see deeply into the content they are studying and fuels new insights about mathematics. Plus, by learning how to make increasingly sophisticated comparisons, students become more adept at avoiding the most common pitfalls of mathematical learning. Comparisons help students:

* See the “invisible” (e.g., closure of sets under operations, mathematical structures).
* Clarify the “confusable” (e.g., expressions vs. equations, permutations vs. combinations).
* Notice the “neglectable” (e.g., divisibility by a variable that could have a zero value).

## Overview

The human brain is hard wired to make comparisons. Compare & Contrast draws its power from this natural cognitive activity, but it also refines students’ comparative skills by teaching them a four-phase process for making quality com­parisons. First, students use a set of clear criteria to describe each item or concept separately. Then, students use a Comparison Organizer to draw out the similarities and differences between the two concepts. Next, students use their comparisons to form conclusions and explore the causes and effects behind key similarities and differences. Finally, students apply their learning by solving a new problem or completing a task.

## Building Common Core Thinking

Compare & Contrast develops students’ comparative thinking and analysis skills in these four areas: description, comparison, conclusion, and application. Compare & Contrast supports the following Standards for Mathematical Practice (MP):

* ***(MP 1) Sense –*** analyzing the givens, constraints, and relationships between problems.
* ***(MP 3) Argument –*** comparing arguments, making and justifying conclusions
* ***(MP 7) Patterns –*** stepping back and looking for patterns and similarity or differences

## Steps

1. Select two (or more) related concepts or mathematics problems.

2. Specify criteria for comparison.

3. Provide (or teach students how to create) graphic organizers for describing items and comparing them.

4. Guide students through the four phases of comparison:

**Figure 2.1: Four Phases of Comparison**

|  |  |
| --- | --- |
| **Description** | 1. Establish purpose for the comparison.
2. Identify sources of information.
3. Clarify criteria for describing items.
4. Have students describe the two items using the Description Organizer.
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| **Comparison** | 1. Provide students with a Comparison Organizer.
2. Have students identify key similarities and differences on the organizer.
 |
| **Conclusion** | 1. Ask students to decide if the two items are more alike or more different.
2. Explore and discuss causes/effects of the differences.
3. Help students form generalizations.
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| **Application** | 1. Present students with a new problem or task.
2. Have students apply their learning to the new problem or task.
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## Examples

Compare & Contrast works equally well with concepts and procedures. Let’s start with procedures. Compare & Contrast is a great way to help students discover the thinking and problem-solving demands behind different kinds of mathematical problems. For example, you might have students compare two different kinds of percentage problems:

### Problem 1

Haylee sells kitchen equipment. As part of her salary, Haylee receives 12% commission on her sales. In February, Haylee sold $7,250 worth of kitchen equipment. How much commission should she receive?

### Problem 2

For March, Haylee forgot to check her total sales figures. She received a commission of $825 for the month. How much did Haylee sell?

First, students describe the two problems using clear criteria, as shown in Figure 2.2.

**Figure 2.2: Describing Two Percentage Problems**

**[INSERT FIGURE 2.2 – see separate file]**

Now, with their descriptions complete, students are in a position to conduct a powerful comparison, as shown in Figure 2.3.

**Figure 2.3: Top Hat (Comparison) Organizer for Percentage Problems**

**[INSERT FIGURE 2.3 – see separate file]**

After the conclusion phase, in which students decide if the two problems are more alike or more different and discuss how the differences affect the problem-­solving procedure, you might ask them to apply their new understanding with a task like the following.

To show what you’ve learned, create three new sales-commission problems. One problem should be like Problem 1 and the other should be like Problem 2. Then, since we noticed that Problem 1 asks you to solve for total commission and Problem 2 asks you to solve for total sales, create a new problem that asks you to figure out the rate (or percentage) of commission.

As we have already noted, Compare & Contrast works as well with mathematical concepts as it does with problem-solving processes. The tool is highly flexible and can be used at varying levels of depth. Students may simply be asked to generate as many similarities and differences as possible and quickly draw conclusions. Here are a few examples of this “down and dirty” approach to Compare & Contrast.

### Odd Numbers and Prime Numbers

* Odd numbers are similar to prime numbers because \_\_\_\_\_.
* Odd numbers are different from prime numbers because \_\_\_\_\_.
* Between the two, the more interesting set of numbers is \_\_\_\_\_ because \_\_\_\_\_.

### Expressions and Equations

* Expressions are similar to equations because \_\_\_\_\_.
* Expressions are different from equations because \_\_\_\_\_.
* Between the two, the harder one to work with is \_\_\_\_\_ because \_\_\_\_\_.

### Permutations and Combinations

* Permutations are similar to combinations because \_\_\_\_\_.
* Permutations are different from combinations because \_\_\_\_\_.
* Between the two, the more useful is \_\_\_\_\_ because \_\_\_\_\_.

Of course, concepts also lend themselves to the full, four-phase Compare & Contrast process. By working with students to run two concepts through the four phases or the “full treatment,” you are committing to the development of their reasoning and analytical skills. For example, you might ask students to:

* Describe *congruence* and *similarity* separately, according to these criteria: symbol notation, size of figures, and shape of figures.
* Identify key similarities and differences on a Comparison Organizer.
* Decide if the two concepts are more similar or more different.
* Apply their learning to a task—a brief essay in which students discuss which concept is more useful in real life.