An Explanation of the Decision-Making Process Behind the Structure of the Problem Situation Tables.

The problem situation tables did not originate with the *Mathematize It!* books, nor did they start with the tables in Appendix A in the Common Core Standards (CCSS-M; National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). The Cognitively Guided Instruction project has used the classification of problem types for decades as one of the important pieces of the puzzle of understanding student thinking (Carpenter, Hiebert, & Moser, 1981).

The categories for classifying problem situations semantically, or according to the meaning of the problem itself, originate in a project from the late 1970s that sorted common word problems into categories that were based more on what happened in the problem than on the numbers from the chapter they were taken from (Heller & Greeno, 1979: Riley, Greeno, and Heller, 1984). This was a big idea because textbooks then, as with now, present word problems in chapters according to the *calculations* needed to solve them.

The type of task sometimes referred to as a word problem (a story problem, real-life math, an application problem, etc.) is always designed to support the computation or conceptual idea that organizes the chapter. It's no wonder students are sometimes left to do "number plucking" (SanGiovanni & Milou, 2018) or randomly assigning operations to solve any problem presented with text.

Of course, learning more about the topic complicated our situation—in a good way. The fact that the Common Core standards (CCSS) and the resources from the Cognitively Guided Instruction (CGI) have different versions of the tables also complicated our process (Carpenter, Fennema, Franke, Levi, & Empson, 2014), so we had to make a number of decisions. Here we share our decision-making process with you, describing how our versions of the tables were settled.

ADDITION AND SUBTRACTION TABLES

DECISION #1 *ADD-TO* OR *JOIN*?

The first two rows of the Addition/Subtraction Problem Types table include problem situations that read like a story: there is a beginning, a middle, and an end. They read like a story because they represent actions that make a change in the problem situation. *Change* is the key word. For CGI, Join and Separate are the names for Row 1 and Row 2 of the table. The Common Core Standards elected to use the terms *Add-To* and *Take-From* to represent the same two rows, respectively. *Join* and *Separate* refer to the physical action of merging two separate sets of objects, but *Add-To* and *Take-From* reflect more common usage for addition and subtraction operations in elementary schools. We did not have a preference, electing to use the CCSS version so that we could acknowledge that version while opting to use CGI's name for Row 3.

DECISION #2 *PART-PART-WHOLE* IS A RELATIONSHIP

The third row of the Addition/Subtraction table does not reflect a story narrative as we saw in the first two rows. Instead, the Part-Part-Whole problem situation reflects a relationship, not action (Carpenter & Moser, 1984). As a result, the problem begins with all elements in place, and the student's job is to puzzle out those relationships using pieces with different attributes or with different equations. Because this is always a non-active context (otherwise it would be an active type) we decided that the title *Put-Together/ Take Apart* implied too much agency within the problem. There is no actor putting these parts together in the problem situation: they simply *are*. Part-Part-Whole seemed the more accurate representation. More importantly, it echoes the relationships within fractions that have a similar structure of parts related to a whole (Kieren, 1976).

DECISION #3 THE SPECIAL CASE OF *BOTH ADDENDS UNKNOWN*

We elected to structure the Part-Part-Whole row differently than both the CGI and CCSS tables. The Both Addends Unknown problem structure is unique in that unlike the others where one of the elements is unknown, this problem structure has both of the addends unknown. Because both are unknown, finding the unknowns is a different kind of task. There is an element of Guess-and-Check and an algebraic-like systematicity involved in the task which isn't the same as we see with only a single element unknown. The encourages flexibility, which is also good for both algebraic thinking and fact fluency development (see Champagne, Schoen, & Riddell, 2014). We decided that it is an entirely different problem situation deserving its own space on the table for two reasons. The Mathematizing Sandbox relies on modeling the problem situation, and the modeling works very differently when two values are unknown. It is also different because the Both Addends Unknown task can really be done with any of the problem types, not just for a Part-Part-Whole. For these reasons, we wanted to separate the Both Addends unknown so that it can be an important standalone task and so that it can be expanded to the problem types in the other rows if a teacher wanted to do so.

DECISION #4 NAMING THE MISSING PARTS IN *COMPARISON*

Comparing the CGI version of the Addition/Subtraction problem types table with the CCSS version we see that each identifies the "Difference" as one of the missing elements in a Comparison. As we noted in the chapter, the difference is a quantity that cannot exist, as it is the space between two real quantities. These other two elements are real, and either one can be missing. In the CGI version of the table, the two elements are related to the grammatical structure of the comparison. One element is referred to as the *quantity* while the other is referred to as the *referent*. When a comparison is made, there is a featured quantity and another is being compared to it. The *referent* element is the one that is featured. It exists, linguistically, as a basis for comparison. The *quantity* is the element that is compared to the *referent*. In our experience with introductory professional development sessions on the problem types table, clarifying and understanding the distinction between the *quantity* and the *referent* distracts from the core goal of distinguishing problem situations among the rows. We also were not convinced that the distinction was salient to the reader.

Instead of focusing on a linguistic structure, the CCSS table instead focuses on the value of the quantities in the problem (Fuson, 2013), distinguishing between the greater and lesser value. Because of the complexity of the linguistics structure of the idea of the *referent*, we elected to use the CCSS categories to distinguish between the two elements in a comparison. Since we anticipated expanding the use of the table to all rational numbers, we tested this decision within the context of comparisons using negative values, and still found it to be a viable categorization.

DECISION #5 ANOTHER ADDITION/SUBTRACTION PROBLEM SITUATION?

While researching the origins of the problem tables, we encountered another problem situation that does not appear in any modern version. The *Equalize* problem situation was identified and we considered its relevance (Carpenter & Moser, 1979). An *Equalize* problem situation invites the reader to balance the two quantities give in the problem and make one equal to the other. An *Equalize* problem might look like this:

> *John has 5 pennies. Maria has 9 pennies. How many pennies does John need in order to have the same number of pennies as Maria has?*

While we considered adding an additional row, we decided the equalize problem type was too much like a comparison, something English (1998) also concluded, and didn't include it in the table. However, with recent focus on distinguishing the equal sign as a sign that shows balance between the quantities on either side, and *not* as an operator (Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012), it might be something worth reviving in the future.

MULTIPLICATION AND DIVISION TABLES

DECISION #6 ASYMMETRIC AND SYMMETRIC MULTIPLICATION AND DIVISION

In the Multiplication and Division table there are two distinct, qualitatively different categories (Bell, Greer, Grimison, Mangan, 1989). These categories separate multiplication with like factors as *symmetric* and multiplication with unlike factors as *asymmetric* (Kouba, 1989; Kouba & Franklin, 1993). Formally called an isomorphism of measures, the two factors in an asymmetric problem situation play “distinct roles” (Bell, et. al, 1989, p.434). One factor is the multiplier while the other describes what is iterated. We made this distinction in the K–2 volume in the early discussion of models of odd and even and emphasized it in the discussion of Equal Groups in the 3–5 book. In the 6–8 book the difference between the two types of factors is highlighted by a scale factor and the quantity it is scaling.

DECISION #7 EXPANDING PARTITIVE AND QUOTITIVE DIVISION MODELS TO MULTIPLICATION

In other versions of the multiplication and division table, two models of division are described: *partitive* and *measurement*. The teachers we work with in professional development often are challenged to discern the difference between these two division models because their familiarity with the operation makes it harder to recognize. Instead of focusing on only the division models, we instead looked to the factors themselves to make a meaningful distinction in the table. After all, the difference between the partitive and measurement models of division is also present in the multiplication interpretation of the same problem situations. For example, in a partitive division problem situation the multiplier factor is known— we know how many groups will be made, but we don’t know the measure factor, or how much will be in each share. This version of the table distinguishes between problem situations by whether the measure or multiplier factor is unknown.

DECISION #8 EQUAL GROUPS ARE REALLY A RATE

Ratios and Rates are a topic typically introduced in the sixth grade and explored in depth throughout middle school and even beyond. While researching problem types and the kind of word problems seen in middle school, it was apparent to us that the typical multiplication and division table used for elementary school was missing this key category of problem context (de la Torre, Tjoe, Rhoads, & Lam, 2013; Lamon, 1993). One idea that emerged was that the multiplier factor in an Equal Groups problem situation was technically an example of one of three definitions of *rate*, as a unit-per-unit comparison (Confrey & Smith, 1994). In the equal groups problem that follows, note the language used with the multiplier factor.

> *To prepare for campers, Marianna placed five cots in each cabin. If there are 10 cabins, how many beds will she place?*

Five cots in each cabin is a rate $\frac{\text{5 cots}}{\text{1 cabin}}$, implying that for each additional cabin, 5 additional cots will be needed. This information is less important in the elementary grades because the implied comparison is always to 1 unit, as in 5 cots to 1 cabin. Since students in sixth grade will address other rates (and ratios), the Ratio/Rate relationship carries more importance. In our definition, the Equal Groups problem situation is one variety of the Ratio/Rate problem situation yet still has its own row on the table, to be consistent with other versions of the table. We added the Ratio/Rate row to the Multiplication and Division Problem Types Table for Grades 6–8 to acknowledge its importance in the middle school curriculum and independence from the Equal Groups subcategory.

DECISION #9 *PRODUCT OF MEASUREMENTS*

Comparisons, Ratio/Rate, and Equal Groups problem situations are asymmetric, meaning that one factor has a unit of measure and the other factor is a rate or scale factor. In this version of the problem types table we describe two symmetric problem situations:

the Area/Array and the Combinatorics (also referenced as the Probability Sample Space or Fundamental Counting Principle, or Cartesian Products). This category is also described as the *product of measures* (Bell, Greer, Grimison, & Mangan, 1989). Similarly, the Area/Array row of the table does not account for the potential for representing sets of coordinates arranged in a matrix, nor does it explain the counting principles that underlie the study of permutations and combinations, which can be challenging for students to understand (Anghlieri, 1989). While middle schoolers don't specifically address Cartesian Products in most standards, they do explore representations that effectively generate the products during the study of probability. For this reason, we added this problem situation to the middle school version of the table.

CONCLUSION

Our intention is to focus on problem solving as an investigative, creative process, one that happens in an environment more like a sandbox than one within a problem solving computer algorithm, which is what Heller & Greeno (1979) were attempting to accomplish when the uncovered the semantic distinctions between problems that form the foundation of this book. Part of this investigative process includes exploring many different problem situations and planning lessons more deliberatively to draw from not only the different rows in a table, but also to assure that students have exposure to problems with any of the three elements missing in the problem situation.

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