Instructional Guide for the 3-Task Sequence

Tasks:	Squares, Circle and Spots, Monstrous Counterexample, Squares (revisited)
Goal:	Checking examples is useful to gain an understanding of a relationship and to help develop a conjecture. However, a set of examples alone is not sufficient to ensure that a conjecture will hold for all cases within the domain of a problem. The way to ensure the truth of a conjecture is to develop a proof.
Evidence:	Initially, students will be convinced that the pattern they found for the number of 3 × 3 squares in a 4 × 4 square and a 5 × 5 square will give the correct answer for the number of 3 × 3 squares in a 60 × 60 square. Then they will learn (through work on the Circle and Spots task) that checking just a few cases is not enough, but they may think that trying a few more cases will be enough, or they will begin to wonder what is the "magic" number of cases to check. Through their work on the Monstrous Counterexample, students will come to realize that there is no subset of examples that will assure you that a conjecture is always true. Finally, students will be motivated to return to the Squares Problem and, with the help of the teacher, prove that their equation will indeed work for the 60 × 60 square.

ACKNOWLEDGMENT

This lesson template is based on the activity sequences discussed in the following:

Stylianides, A. J. (2009). Breaking the equation "empirical argument = proof." *Mathematics Teaching*, *213*, 9–14.

Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education*, *40*(3), 314–352.

SQUARES PROBLEM (PART 1)

Task

What is the main activity that students will be working on in this lesson?

Find the number of 3 × 3 squares in a 60 × 60 square.

What are the various ways that students might complete the activity?

1. **Make a table and guess/notice a formula that works for the first few situations.**
2. **Count the squares in a 4 × 4 and a 5 × 5 square, and use a recursive method to reach the answer for a 60 × 60 square.**
3. **Count the squares in a 4 × 4 and a 5 × 5 square, and use a proportional reasoning method to reach the answer for a 60 × 60 square.**
4. **Students may say that only one 3 × 3 square fits in a 4 × 4 if they do not see that the 3 × 3 squares can overlap.**
5. **Students may want to count the number of 3 × 3 squares in a 60 × 60 square.**

Instructional Support

What tools or resources will students have to use in their work that will give them entry to, and help them reason through, the activity?

Task sheet

Graph paper

3 × 3 square (optional)

Colored pencils/markers

What questions might you ask students that will support their exploration of the activity and ***bridge*** *between* ***what they did*** *and* ***what you want them to learn*** *(the two shaded boxes)?*

1. Ask the students how many 3 × 3 squares they believe will fit into a 60 × 60 square. Then ask how they got their number (correct or not). Ask how confident they are in the number being correct. Why do they think they are right?
2. The recursive method will produce a correct solution for 60 × 60, but it is not practical. Can you find another way?
3. Ask the students to first guess the number of 3 × 3 squares that would fit in a 6 × 6 square. Then have them actually count the number of 3 × 3s and compare this solution to what they get when they use the proportional method. Then have the students check the proportion 5 × 5 to 60 × 60 and 6 × 6 to 60 × 60. They will not match 9 x 12 = 108 and 16 × 10 = 160. Proportional reasoning also fails at the 4 × 4 case.
4. Ask the students if there are other squares in the 4 × 4 *if* it is okay for the 3 × 3s to overlap.
5. Acknowledge that this way would work, but is this the most efficient way? What if someone asked about an even bigger square?

Learning Goals (Residue)

What understandings will students take away from this activity?

Students will identify a pattern and will become aware that their way of justifying the pattern is by checking a few examples. [The following activities will help them begin to question this approach to validating patterns.]

Evidence

What will students say, do, produce, etc., that will provide evidence of their understandings?

Students should agree that they found the number of 3 × 3 squares in a 60 × 60 square by subtracting 2 from 60 and squaring it $(60 - 2)^2$. Students should then be asked individually whether they are confident or unsure about this finding. The dominant/most common response should be made public by the teacher.

CIRCLE & SPOTS

Task

What is the main activity that students will be working on in this lesson?

Circle and Spots problem

(Suggest that students start with one dot instead of zero dots or just that they do not include zero dots.)

What are the various ways that students might complete the activity?

1. **Students may only draw two lines for three dots or not know that the lines can intersect.**
2. **Students might make a table and begin to guess (using a calculator) a formula that relates to the first few cases.**
3. **Students may become concerned about how close together or far apart the dots are placed around the circle.**
4. **Students may attempt to draw 15 dots, connect them, and attempt to count the regions formed.**

Instructional Support

What tools or resources will students have to use in their work that will give them entry to, and help them reason through, the activity?

Task sheet

Paper with circles

Ruler

Be prepared to show or draw a six-spot circle that yields 31 regions.

Calculators not recommended

What questions might you ask students that will support their exploration of the activity and ***bridge*** *between* ***what they did*** *and* ***what you want them to learn*** *(the two shaded boxes)?*

1. Remind the students that lines need to connect all of the dots. This is a situation in which telling the students will not take away from your goals of the lesson.
2. Ask students what is happening to the number of regions as a new dot is added. Then ask if they believe their pattern will stay true no matter how many dots are added. Then have them make a guess about six dots and then check by drawing the lines.
3. Perfectly spaced dots will produce only 30 regions for six dots, but this will not happen without a compass and large enough circles. The dots need to be far enough apart to draw a line and see a region along the circumference. Again, telling students this information does not subtract from the goal.
4. The students will not be able to complete this solution because the figure becomes extremely messy.

Learning Goals (Residue)

What understandings will students take away from this activity?

Students will learn that some patterns do not hold true beyond a specific number of cases. However, some students will believe that they just need to check more cases to be sure that a formula holds, while others will be unsure about the number of cases that ought to be checked in order to be certain that a generalization will always work.

Evidence

What will students say, do, produce, etc., that will provide evidence of their understandings?

Most students will suggest that the number of regions doubles every time a new dot is added to the circumference. If no one notices, ask them to check what happens with six dots.

After the counterexample is discussed, ask students what the Circle and Spots problem taught them. Some students may suggest that more cases need to be checked to determine that a pattern is true. Others might suggest choosing to check larger numbers to see if the patterns continue as expected.

MONSTROUS COUNTEREXAMPLE

Task

What is the main activity that students will be working on in this lesson?

Monstrous counterexample

What are the various ways that students might complete the activity?

Use a calculator or spreadsheet to test various values for *n*. Then take the square root to verify that the result is not a square number.

Students will become convinced that the expression will always work.

Instructional Support

What tools or resources will students have to use in their work that will give them entry to, and help them reason through, the activity?

A slide/overhead of the expression

Calculators or spreadsheets

What questions might you ask students that will support their exploration of the activity and ***bridge*** *between* ***what they did*** *and* ***what you want them to learn*** *(the two shaded boxes)?*

This exercise could be more leading. The point is to help students see that no matter how many examples they check or how they choose those examples, a pattern can fail in a case they didn't check. Have the students test a sufficient number of cases and check with group members.

Learning Goals (Residue)

What understandings will students take away from this activity?

There is no one defined subset of cases that can be checked to *prove* a conjecture is always true.

Evidence

What will students say, do, produce, etc., that will provide evidence of their understandings?

Students will start expressing a distrust in any kind of empirical arguments. Some of them may even express a distrust in patterns. They are now ready to hear about the role of proof in mathematics.

SQUARES PROBLEM (PART 2)

Task

What is the main activity that students will be working on in this lesson?

Revising their argument to show that the pattern holds for a 60 × 60 square.

What are the various ways that students might complete the activity?

1. Students may not be sure what to do to make sure that the pattern is true for a 60 × 60 square.
2. A student explains how she got to 58^2 or $(60 - 2)^2$ in relation to how she counted the 4 × 4, 5 × 5, and 6 × 6 squares.

Instructional Support

What tools or resources will students have to use in their work that will give them entry to, and help them reason through, the activity?

Task sheet

Graph paper

Colored pencils/markers

*What questions might you ask students that will support their exploration of the activity and **bridge** between **what they did** and **what you want them to learn** (the two shaded boxes)?*

1. Look to see if students chose a systematic approach to counting the number of 3 × 3 squares. Ask them to show how they counted the 3 × 3 squares in the 4 × 4, 5 × 5, and 6 × 6 squares. If their counting is unsystematic, then ask them to choose a single systematic method for counting. Ask them to try 7 × 7. At this point, they should know that there are 25 squares, but how would they find this number without one-by-one counting? Students may figure this out in various ways (see possible solutions), but each student should stay systematic with his or her particular method.
2. This is a main target for this task. It is not expected that students explain the general case. Ask the student to write this up on flip chart paper to see if her classmates are convinced that the 60 × 60 case is proven.

Learning Goals (Residue)

What understandings will students take away from this activity?

Address the possible misconception that one can never trust patterns, and also get an image of what it means to produce a proof.

Evidence

What will students say, do, produce, etc., that will provide evidence of their understandings?

Explanations in which students connected their solution for the 60 × 60 square with how they counted the number of 3 × 3 squares in the 4 × 4, 5 × 5, and 6 × 6 squares.