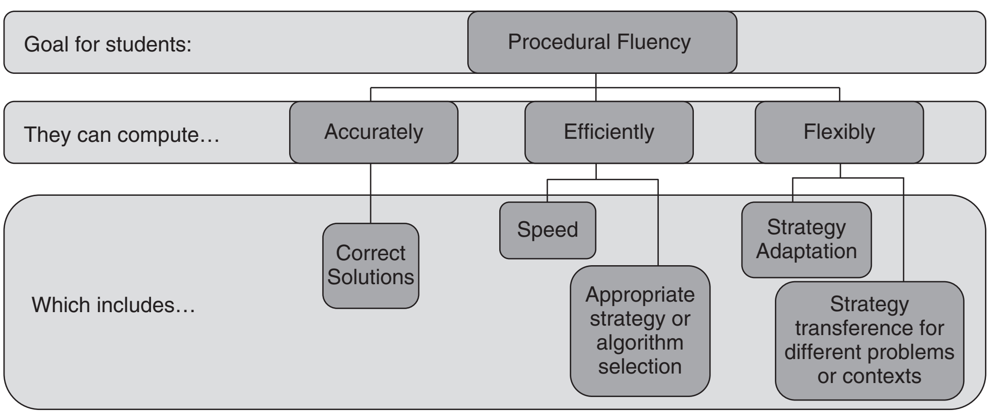
Overview of Content Knowledge and Worthwhile Tasks

The mathematical knowledge of teachers impacts their students' achievement, in particular the **mathematical knowledge for teaching (MKT**; Hill, Rowan, & Ball, 2005). And this specialized knowledge develops over time as teachers analyze curriculum and possible learning trajectories for students, consider common errors or misconceptions, and implement worthwhile tasks. Therefore, time spent focused on developing mathematical knowledge connected to what a teacher is teaching is time well spent! While there are many ways to "get at" MKT, here we zoom in on two important ideas: (1) understanding the relationship between and the importance of conceptual and procedural knowledge and (2) considering what makes a task worthwhile and how to adapt a task to make it worthwhile.

Developing Fluency

The fact that "Build procedural fluency from conceptual understanding" is one of eight Teaching Practices is an indication of the critical importance of these two knowledge domains. Rather than pit procedural knowledge against conceptual knowledge, both can be thought of as on a continuum from weak to strong (Star, 2005). This is a much more useful way to present conceptual and procedural knowledge because, in fact, procedural fluency requires depth of knowledge in *both*. Too often, **fluency** instruction is limited to developing facility with a single procedure for a particular topic. But **procedural fluency** includes four components: accuracy, efficiency, appropriate **strategy** selection, and flexibility (NGA & CCSSO, 2010; NRC, 2001). These final two components are really key to actual fluency and must be included in fluency instruction. As illustrated in Figure 3.1, efficiency is not only about speed but also about strategy selection. For example, how might you efficiently solve the problem 3,005–1,998? A mental counting-up (or back) strategy is more efficient than applying the standard **algorithm** (that requires regrouping over zeros).

Figure 3.1 **Procedural Fluency Diagram**



Source*: Bay-Williams, J. M., and Stokes Levine, A. (2017). "The Role of Concepts and Procedures in Developing Fluency." In D. Spangler & J. Wanko (Eds.),* Enhancing Professional Practice With Research Behind Principles to Actions. *Reston, VA: NCTM.*

Strategy selection and flexibility require high-level thinking. Bloom's **Taxonomy** provides a useful framework to consider low-level to high-level thinking (see Chapter 5). Fan and Bokhove (2014) organized Bloom's Taxonomy and computational actions into three levels of cognition (see Figure 3.2). Level 1 computational actions are low level on Bloom's Taxonomy (*Remember*). Notice that these actions, however, have dominated mathematics teaching and learning, sometimes being the sole focus of worksheets, textbook lessons, and classroom discussions. Levels 2 and 3 are high-level computational actions. A key distinction between Levels 2 and 3 is that Level 2 focuses on understanding one particular procedure, and at Level 3, the focus is on contrasting and comparing several procedures (Fan & Bokhove, 2014).

Figure 3.2 **Mapping Fluency Thinking to Bloom's (Revised) Taxonomy**

A lower level supports a higher level and vice versa.

Cognitive Levels (Fan & Bokhove, 2014)	*Bloom's (Revised) Elements*	*Related Actions With Procedures*
Level 1. ***Knowledge and Skills***	1. Remember	Tell the steps of a procedure. Carry out steps in a straightforward situation.
Level 2. ***Understanding and Comprehension***	2. Understand	Describe why a procedure works. Apply procedure to complex problems.
	3. Apply	
	4. Analyze	
Level 3. ***Evaluation and Construction***	5. Evaluate	Compare different algorithms. Judge efficiency of an algorithm. Construct new algorithms (strategies). Generalize when a procedure works.
	6. Create	

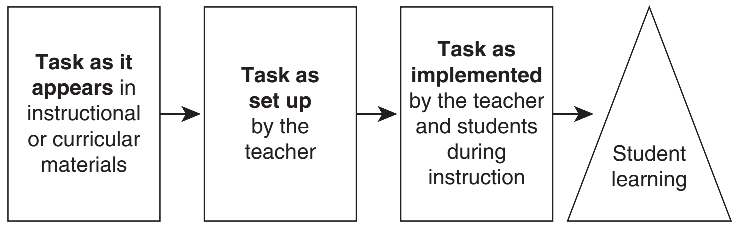
Source*: Based on Fan, L., and Bokhove, C. (2014). "Rethinking the Role of Algorithms in School Mathematics: A Conceptual Model With Focus on Cognitive Development."* ZDM International Journal on Mathematics Education, 46*(3), 481–492.*

Historically, school mathematics learning has had an overemphasis on demonstrating one procedure or algorithm for a particular problem type and having students practice it (Level 1). This results in weak procedural knowledge and weak conceptual knowledge. Mathematics tasks and classroom instruction must spend significant time on Levels 2 and 3 for students to develop procedural fluency. Consider how a topic such as division of fractions can look across these levels, using the example $4\frac{1}{2} \div \frac{3}{4}$. At Level 1, a student might be shown and asked to remember the "invert and multiply" algorithm. At Level 2, students are able to describe the meaning of the operation. A student might say, "I am trying to find how many $\frac{3}{4}$ are in $4\frac{1}{2}$," or might describe a situation, "This is like needing $\frac{3}{4}$ of a yard of fabric to make each apron, and finding how many aprons can be made from $4\frac{1}{2}$ yards of fabric." And they can describe why an algorithm works. For example, noting that the reason you multiply by 4 is that you can first count how many fourths in $4\frac{1}{2}$. Since each whole has 4 fourths, you multiply. And since you are finding groups of three (for $\frac{3}{4}$), you divide by 3. At Level 3, students are asked to consider various procedures and expected to flexibly select one that fits the numbers. In this case, they may invert and multiply, or they may use a counting-up strategy: Two aprons would take $1\frac{1}{2}$ yards, so four aprons would take 3 yards, so six aprons would take $4\frac{1}{2}$ yards. Or they may mentally think of $4\frac{1}{2}$ as 18 fourths and then divide 18 by 3 to equal 6. One way to focus on Level 3 is to analyze a worked example or compare two student-worked examples, a practice that can improve student achievement (Renkl, 2014; Star & Verschaffel, 2017). Worked examples can be correct, incorrect, or incomplete (i.e., a student gets "stuck"), and students are asked to analyze the strategy, find the error, or help complete the task. Notice how conceptual knowledge supports and connects to procedural knowledge when working at Levels 2 and 3 and that the result is a deeper understanding of the topic.

Worthwhile Tasks

The development of deep-content knowledge occurs when students have the opportunity to engage in worthwhile mathematics tasks—worthwhile because they provide students an opportunity to apply important mathematical properties, make connections, and think at a high level. To provide these opportunities to learn (OTL) requires finding such a task, then maintaining the rigor or high-level thinking of the task in planning and in teaching. This is effectively described in the Mathematical Tasks Framework (Smith & Stein, 1998; see Figure 3.3).

Figure 3.3 **Mathematical Tasks Framework**



Source*: Stein, M. K., and Smith, M. S. (1998). "Mathematical Tasks as a Framework for Reflection: From Research to Practice."* Mathematics Teaching in the Middle School, 3*(4), 268–275. Reprinted with permission. Copyright by the National Council of Teachers of Mathematics. All rights reserved.*

The goal for mathematics teachers is to maintain the level of cognitive demand in a task rather than "help" students in ways that lower the level of demand. Numerous studies have found that increased exposure to cognitively challenging tasks and extended engagement with high-level cognitive demands increases students' learning of mathematics (e.g., Hiebert & Wearne, 1993; NCES, 2003; Stein & Lane, 1996).

To start, worthwhile tasks should feel worthwhile to a student. That means the context and/or the mathematical questions posed must be engaging. Relevant contexts serve the purpose of providing a concrete link to the abstract mathematical ideas, as well as seeing how mathematics can be used to learn about and to serve the school and/or community. Using everyday situations can increase student participation, increase student use of different problem strategies, and help students develop a productive disposition (Tomaz & David, 2015).

Second, worthwhile tasks also have **multiple entry points**, meaning that the task can be approached in a variety of ways and has varying degrees of challenge within it. Having multiple entry points serves various purposes. First and foremost, such tasks better meet the needs of diverse learners because students can select an approach based on their prior experiences and knowledge. Second, having multiple entry points opens up the opportunity to compare and evaluate strategies for solving the problem. This provides an opportunity to see relationships among representations, as well as see different ways to symbolically solve a problem and discuss efficient approaches. Third, having multiple entry points provides more insights into student thinking and understanding, providing important and useful formative assessment data.

Worksheets and textbook problems can often be closed, low level, and uninteresting, but there are often "tweaks" that can be made to the instructions or to a task that can change it into a much more interesting and higher-level thinking activity. Boaler (2016) provides suggestions on adapting procedural tasks with such a goal in mind. In addition to multiple entry points, these ideas include the following:

- *Grow the task:* Change a task from a single computation to finding possibilities. For example, rather than add 23 + 15, find numbers that result in 50.
- *Invite multiple ways:* Ask students to use multiple strategies and representations.
- *Add a visual requirement:* Ask students to show two different visuals or connect between representations (e.g., story and equation).
- *Reason and convince:* Ask students to create convincing arguments and to expect the same from their peers.